Square Dancing 12

I BET I CAN guess your favorite math subject in high school.

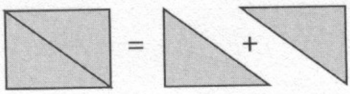
It was geometry.

So many people I've met over the years have expressed affection for that subject. Is it because geometry draws on the right side of the brain, and that appeals to visual thinkers who might otherwise cringe at its cold logic? Maybe. But some people tell me they loved geometry precisely because it *was* so logical. The step-by-step reasoning, with each new theorem resting firmly on those already established—that's the source of satisfaction for many.

But my best hunch (and, full disclosure, I personally love geometry) is that people enjoy it because it *marries* logic and intuition. It feels good to use both halves of the brain.

To illustrate the pleasures of geometry, let's revisit the Pythagorean theorem, which you probably remember as $a^2 + b^2 = c^2$. Part of the goal here is to see why it's true and appreciate why it matters. Beyond that, by proving the theorem in two different ways, we'll come to see how one proof can be more elegant than another, even though both are correct.

The Pythagorean theorem is concerned with right triangles—meaning those with a right (90-degree) angle at one of the corners. Right triangles are important because they're what you get if you cut a rectangle in half along its diagonal:

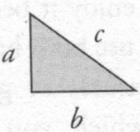


And since rectangles come up often in all sorts of settings, so do right triangles.

They arise, for instance, in surveying. If you're measuring a rectangular field, you might want to know how far it is from one corner to the diagonally opposite corner. (By the way, this is where geometry started, historically—in problems of land measurement, or measuring the earth: *geo* = "earth," and *metry* = "measurement.")

The Pythagorean theorem tells you how long the diagonal is compared to the sides of the rectangle. If one side has length a and the other has length b, the theorem says the diagonal has length c, where

$$a^2 + b^2 = c^2.$$



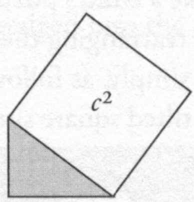
For some reason, the diagonal is traditionally called the hypotenuse, though I've never met anyone who knows why. (Any Latin or Greek scholars?) It must have something to do with the diagonal subtending a right angle, but as jargon goes, "subtending" is about as opaque as "hypotenuse."

Anyway, here's how the theorem works. To keep the numbers simple, let's say $a = 3$ yards and $b = 4$ yards. Then to figure out the unknown length c, we don our black hoods and intone that c^2 is the sum of 3^2 plus 4^2, which is 9 plus 16. (Keep in mind that all of these quantities are now measured in square yards, since we squared the yards as well as the numbers themselves.) Finally, since $9 + 16 = 25$, we get $c^2 = 25$ square yards, and then taking square roots of both sides yields $c = 5$ yards as the length of the hypotenuse.

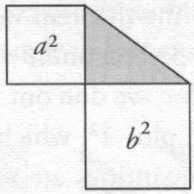
This way of looking at the Pythagorean theorem makes it seem like a statement about lengths. But traditionally it was viewed as a statement about *areas*. That becomes clearer when you hear how they used to say it:

> The square on the hypotenuse is the sum of the squares on the other two sides.

Notice the word "on." We're not speaking of the square *of* the hypotenuse—that's a newfangled algebraic concept about multiplying a number (the length of the hypotenuse) by itself. No, we're referring here to a square literally sitting *on* the hypotenuse, like this:

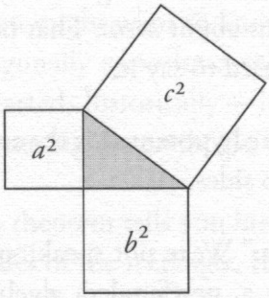


Let's call this the large square, to distinguish it from the small and medium squares we can build on the other two sides:



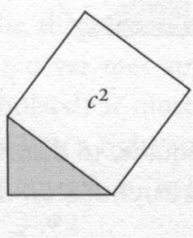
Then the theorem says that the large square has the same area as the small and medium squares combined.

For thousands of years, this marvelous fact has been expressed in a diagram, an iconic mnemonic of dancing squares:



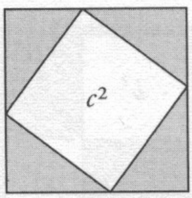
Viewing the theorem in terms of areas makes it a lot more fun to think about. For example, you can test it—and then eat it—by building the squares out of many little crackers. Or you can treat the theorem like a child's puzzle, with pieces of different shapes and sizes. By rearranging these puzzle pieces, we can prove the theorem very simply, as follows.

Let's go back to the tilted square sitting on the hypotenuse.



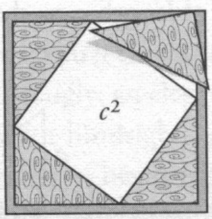
At an instinctive level, you should feel a bit disquieted by this image. The square looks potentially unstable, like it might topple or slide down the ramp. And there's also an unpleasant arbitrariness about which of the four sides of the square gets to touch the triangle.

Guided by these intuitive feelings, let's add three more copies of the triangle around the square to make a more solid and symmetrical picture:



Now recall what we're trying to prove: that the tilted white square in the picture above (which is just our earlier large square—it's still sitting right there on the hypotenuse) has the same area as the small and medium squares put together. But where are those other squares? Well, we have to shift some triangles around to find them.

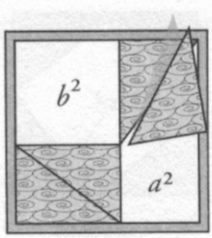
Think of the picture above as depicting a puzzle, with four triangular pieces wedged into the corners of a rigid puzzle frame.



In this interpretation, the tilted square is the empty space in the middle of the puzzle. The rest of the area inside the frame is occupied by the puzzle pieces.

Now let's try moving the pieces around in various ways. Of course, nothing we do can ever change the total amount of empty space inside the frame—it's always whatever area lies outside the pieces.

The brainstorm, then, is to rearrange the pieces like this:

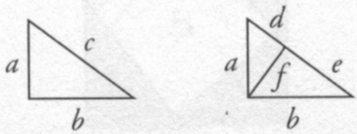


All of a sudden the empty space has changed into the two shapes we're looking for—the small square and the medium square. And since the total area of empty space always stays the same, we've just proven the Pythagorean theorem!

This proof does far more than convince; it *illuminates*. That's what makes it elegant.

For comparison, here's another proof. It's equally famous, and it's perhaps the simplest proof that avoids using areas.

As before, consider a right triangle with sides of length a and b and hypotenuse of length c, as shown below on the left.



Now, by divine inspiration or a stroke of genius, something tells us to draw a line segment perpendicular to the hypotenuse and down to the opposite corner, as shown in the triangle on the right.

This clever little construction creates two smaller triangles inside the original one. It's easy to prove that all these triangles are similar—which means they have identical shapes but different sizes. That in turn implies that the lengths of their corresponding parts have the same proportions, which translates into the following set of equations:

$$\frac{a}{f} = \frac{b}{e} = \frac{c}{b}$$

$$\frac{a}{d} = \frac{b}{f} = \frac{c}{a}$$

We also know that

$$c = d + e$$

because our construction merely split the original hypotenuse of length c into two smaller sides of lengths d and e.

At this point you might be feeling a bit lost, or at least unsure of what to do next. There's a morass of five equations above, and we're trying to whittle them down to deduce that

$$a^2 + b^2 = c^2.$$

Try it for a few minutes. You'll discover that two of the equations are irrelevant. That's ugly; an elegant proof should involve nothing superfluous. With hindsight, of course, you wouldn't have listed those equations to begin with. But that would just be putting lipstick on a p . . . (the missing word here is "proof").

Nevertheless, by manipulating the right three equations, you can get the theorem to pop out. See the notes on page 272 for the missing steps.

Would you agree with me that, on aesthetic grounds, this proof is inferior to the first one? For one thing, it drags near the end. And who invited all that algebra to the party? This is supposed to be a geometry event.

But a more serious defect is the proof's murkiness. By the time you're done slogging through it, you might believe the theorem (grudgingly), but you still might not *see* why it's true.

Leaving proofs aside, why does the Pythagorean theorem even matter? Because it reveals a fundamental truth about the nature of space. It implies that space is flat, as opposed to curved. For the surface of a globe or a bagel, for example, the theorem needs to be modified. Einstein confronted this challenge in his general theory of relativity (where gravity is no longer viewed as a force, but rather as a manifestation of the curvature of space), and so did Bernhard Riemann and others before him when laying the foundations of non-Euclidean geometry.

It's a long road from Pythagoras to Einstein. But at least it's a straight line . . . for most of the way.